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On another type of ducks in the FitzHugh-Nagumo equation

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Abstract: This paper shows that if the constant r in the FitzHugh-Nagumo equation tends to zero, then the winding number for one of ducks tends to infinity and the number for the other duck remains infinitesimal. As a result we are led to the conclusion that there exists a duck which is not S^1 .

1.Introduction.

The space-clamped FitzHugh-Nagumo (FHN) equation (1.1) is described as follows:

$$dv/dt=-\rho(v)-w+ I, \qquad (1.1)$$
$$dw/dt=b(v-rw), \quad \rho(v)=v(v-1)(v-a),$$

where $0<a<1/2$, b and r are positive constants. Each variable has any physical meaning as follows:

v(t): the potential difference at the time t across the membrane of the axon,

w(t): a recovery current which is often taken to be the sum of all ion flows, and

I: an independent variable as a bifurcation parameter which varies very slowly. The constant r is restricted so that

$$1/r>(a^2-a+1)/3. \qquad (1.2)$$

The constant b is very small constant. Thus, we assume that

$$b=c\varepsilon \qquad (1.3a)$$

and

$$I=I_0+\varepsilon t. \qquad (1.3b)$$

In the asymptotic expansion for the solution of (1.1), Baer et al. [1] analyzed the delay phenomena when $\varepsilon= O(b^{3/2})$ and $I=I_0+\varepsilon t$. When $\varepsilon=O(b)$, there is a difficulty that the uniformity of the solution is destroyed in the expansion. In other words, there exists the singularity in this dynamical bifurcation problem. We have already provided an answer for this problem: there exist S^1 ducks in the FHN equation [3].

In this paper, the following theorem1 and theorem2 are provided. As a result, we can obtain theorem3: there exists another type of ducks in the equation.

Theorem1.

Let $\mu_1<\mu_2<0$ hold. If the constant r tends to zero, then the winding number $N\gamma\,\mu_1(\psi_1)$ for a duck $\gamma\,\mu_1$ tends to infinity.

Theorem2.

If $\psi_i(i=1,2)$ does not contain the parts of the solutions which jump from the repulsive region to the attractive one, then in the local model of the FHN equation with the condition (1.2), the lower bound of $|N\gamma\,\mu_1(\psi_1)|$ is 1 and $|N\gamma\,\mu_2(\psi_2)|$ is infinitesimal.

Theorem3.

If the constant r is sufficiently small under the condition (1.2), then there exists a duck which is not S^1.

2.Notations and Definitions.

The definitions of the ducks, the pseudo singular point and its saddle, node point follow as [3]. Especially important notations and definitions using in this paper are denoted as follows:

Definition2.1.

We call a point P is a δ-micro-galaxy of $E\subset R^3$, if the distance from P to E is less than $\exp(-n/\delta)$ where n is some positive integer.

This definition is based on the fact: if δ is fixed arbitrarily, then the distance from a duck $\gamma(t)$ to $\gamma\mu(t)$ is within $\exp(-n/\delta)$ in some neighbourhood of the pseudo singular point.

Definition2.2.

A duck is called long if $\psi(t)$ exists an infinitesimally small neighbourhood of the constrained surface S when t tends to infinity.

Definition2.3.

A solution $\psi(t)$ is called S^1 at α if there exists a real number β such that

$$(\psi(x)-\psi(y))/(x-y)\approx\beta \tag{2.1}$$

for any x, y $(x\approx\alpha,\ y\approx\alpha)$. A duck is called an S^1 duck if it is S^1 in some neighbourhood of the pseudo singular point.

Definition2.4.

The winding number $N\gamma\,\mu(\psi)$ of ψ is defined as follows:

$$N\gamma\,\mu(\psi)=(1/2\pi)\int_{\psi}d\theta \tag{2.2}$$

where ψ is one of the ducks which is contained partially in the δ-micro-galaxy of $\gamma\mu$.

Definition2.5.

A solution $\psi(t)$ has a jump if the shadow of $\psi(t)$ contains a vertical segment.

We assume that ψ is not long. Then ψ is away from the repulsive part of the surface S at a limited t when t tends to ∞ and away from the attractive one of S when t tends to $-\infty$.

It can be proved that

(1) if ψ is not long, the standard part of the number $N\gamma\,\mu(\psi)$ is an integer,

(2) if ψ has k jumps, then $2 \leq k \leq 4$. See [2].

3. The proof of the theorems.

By changing the coordinates w=X, I=Y and v=Z, the system (1.1) becomes as follows:

$$\begin{aligned} dX/dI &= c(Z-rX), \\ dY/dI &= 1, \\ \varepsilon\, dZ/dI &= (-\rho(Z)-X+Y). \end{aligned} \tag{3.1}$$

Furthermore, applying several transformations and putting I=t, the following system (3.2) is obtained as a "local model" (conf. [3]).

$$\begin{aligned} dX/dt &= pY+qZ \quad +\xi(X,Y,Z), \\ dY/dt &= 1 \qquad\quad +\eta(X,Y,Z), \\ \delta\, dZ/dt &= -(Z^2+X)+\zeta(X,Y,Z), \end{aligned} \tag{3.2}$$

where $p=(-1)^i cr(a^2-a+1)^{1/2}$ (i=1,2),

$q=c$, $\delta=\varepsilon/\alpha^2$.

Here $\xi(X,Y,Z)$, $\eta(X,Y,Z)$ and $\zeta(X,Y,Z)$ are infinitesimal when X, Y and Z are limited and α is any small constant. Choosing $p=cr(a^2-a+1)^{1/2}$ and $c>8r(a^2-a+1)^{1/2}$ so that the system (3.2) has a pseudo singular node point, the explicit duck solution $\gamma\,\mu_i(t)$ is obtained as follows:

$$\gamma\,\mu_i(t)=(-\mu_i^2 t^2-\delta\,\mu_i,\ t,\ \mu_i t) \quad (i=1,2). \tag{3.3}$$

Here μ_i(i=1,2) are the solutions of the following characteristic equation with respect to the linearized equation on the surface $-(Z^2+X)=0$ in (3.2):

$$2\mu^2+c\mu+cr(a^2-a+1)^{1/2}=0. \tag{3.4}$$

The Hermite equations associated with $\gamma\,\mu_i$(i=1,2) in (3.3) are the following:

$$\delta\, d^2Z/d\tau^2-\tau\, dZ/d\tau+K_i Z=0 \quad (i=1,2), \qquad \tau=\alpha t, \tag{3.5}$$

where $K_1=1+\mu_2/\mu_1$, $K_2=1+\mu_1/\mu_2$. See[2]. Let ψ_i for $\gamma\,\mu_i$(i=1,2) be one of the ducks. In the case which ψ_i is not long and $\mu_2<\mu_1<0$, Benoit gives the following results:

(1) If the duck ψ_1 has 2 jumps, then

$N\gamma\,\mu_1(\psi_1)\approx-[K_1/2]$,

(2) if the duck ψ_2 has 2 jumps, then

$N\gamma\,\mu_2(\psi_2)\approx 0$.

In the case 3 and 4 jumps, they are treated in the same manner.

Proof of theorem1.

The value of the index K_1 is as follows:

$K_1=[1+\mu_2/\mu_1]$
$=[1+(c+(c^2-8cr(a^2-a+1)^{1/2}))^{1/2})/(c-(c^2-8cr(a^2-a+1)^{1/2})^{1/2})]$
$=[(c+(c^2-8cr(a^2-a+1)^{1/2})^{1/2})/4r(a^2-a+1)^{1/2}]$. (3.6)

For any fixed a and c which satisfy the conditions, the value of K_1 is monotone decreasing with respect to r. As r restricted (1.2) tends to 0, K_1 tends to infinity. Therefore $N\gamma\mu_1(\Psi_1)$ tends to infinity.

Proof of theorem2.

From the condition (1.2), $0<r<3/(a^2-a+1)$. On the other hand, K_1 is continuous function of r. Furthermore, note that $A=(a^2-a+1)^{1/2}\geq 3^{1/2}/2$. When $r=3/A^2$,

$$K_1=(2c^2 A-24c+2c(c^2A^2-24cA)^{1/2}/24c \qquad (3.7)$$

holds. Then, considering the above inequality, we get

$$K_1\geq(3^{1/2}c^2-24c+2c(3c^2/4-12c3^{1/2})^{1/2}/24c$$
$$=3^{1/2}(c+(c^2-16\cdot 3^{1/2}c)^{1/2}/24-1. \qquad (3.8)$$

As $c\geq 16\cdot 3^{1/2}$, we conclude that

$K_1\geq 2$, therefore the lower bound of $|N\gamma\mu_1(\Psi_1)|$ is 1. From theorem1, it is obvious that $|N\gamma\mu_2(\Psi_2)|$ is infinitesimal.

Theorem2 ensures the feasibility of evaluating the winding numbers under the condition (1.2), when r tends to zero and to $3/A^2$.

Proof of theorem3.

From theorem2, even if r tends to 0, each winding number does not coincide. It can be proved that if J=[AB] is a connected segment in R^3 and any solution starting at J is not long, it has the same winding number. Theorem1 relates this results. If each duck ($\gamma\mu_1$, $\gamma\mu_2$) starting at A(or B) and passing the pseudo singular node point satisfies S^1, then the solutions starting at J belongs to one of the families of the two ducks. Therefore, the above connected segment J could not be constructed. In fact, there is a proper subset $[CD]\subset J$ such that the solution starting at [CD] is not long and the two solutions passes C and D are ducks, that is, there exists a duck which does not be S^1.

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